A NEW RECTANGULAR FINITE ELEMENT FOR PLANE ELASTICITY ANALYSIS

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ABSTRACT

A simple rectangular element having 2d.o.f/node at each node is developed. This element is based on the strain approach. From some numerical examples, by using the concept of static condensation it is concluded that the exact solutions can be obtained. This element is nonconforming but satisfies the patch test and produces results which are acceptable within practical engineering accuracy even when few elements are employed.

KEY WORDS: Rectangular element, strain approach, static condensation, elasticity problems.

1 INTRODUCTION

The development of finite elements for general plane elasticity problems occupied a prominent position in the early work on the matrix displacement method of analysis. Attention was therefore focused on the development of more sophisticated elements based on the strain element Sabir et al [1, 2, 3, and 4]. Several models such as rectangular elements were developed, among them the elements of Sabir et al [4] SBRIE and SBRIE2. The first element is based on linear variation of direct strains and constant shearing strain. The second is based on linear variation of all three strain components. These elements produce rapid convergence of deflections as well as stresses. A further progress into the development of plane stress elements based on the strain approach is due to Belarbi [5, 6, 7 and 8].

In this paper the shape function for a rectangular element having two degrees of freedom at each of the four corner nodes is developed using the strain approach. However any singularity is eliminated by the use of local axes optimally oriented. This element is nonconforming but satisfies the patch test and produces results which are acceptable within practical engineering accuracy even when few elements are employed.

2 CONSTRUCTING THE COMPONENT STIFFNESS MATRICES

The above consideration will lead to an element requiring ten d.o.f, each of the four corner nodes has the two essential d.o.f, in addition, an internal node is also used fig.1

The assumed strains are:

\[
\begin{align*}
\epsilon_x &= a_1 y + a_4 y - a_5 x - \alpha a_{10} x \\
\epsilon_y &= -a_3 y + a_6 + a_7 x - \alpha a_{10} y \\
\gamma_{xy} &= a_8 + a_9 x + a_{10} y
\end{align*}
\] (1)

![Figure 1: Co-ordinates and nodal points for the rectangular R4SB2 element](image)
By integrating equations (1) and adding the condition of rigid body movement (R4SB2) we obtain:

\[
\begin{align*}
U &= a_1 - a_2 x + a_3 x y - a_4 x \frac{1}{2}(x^2 + y^2) + a_5 y - a_6 y \frac{1}{2}(\alpha x^2 - y^2) \\
V &= a_7 + a_8 x - a_9 x \frac{1}{2}(x^2 + y^2) + a_{10} y + a_{11} x + a_{12} x \frac{1}{2}(\alpha y^2 - x^2)
\end{align*}
\]

(2)

\(\alpha\) coefficient of condensation, in our case \(\alpha = 150\)

Where \(U\) and \(V\) are the displacements in X and Y direction respectively, rigid body movement is represented by the terms associated with the constants \(a_1, a_2\) and \(a_3\) while the straining of the element is represented by the remaining constants. This element correctly represents the R4SB2 and constant strain states.

The stiffness matrix is derived without using any tricks, which implies that it is obtained using exact and not reduced integration.

\[
[K_e] = [A^{-1}]^T [K_0] [A^{-1}]
\]

(3a)

\[
[K_0] = \int \int [Q]^T [D][Q] dx dy
\]

(3b)

With

\[
[Q] = \begin{bmatrix}
0 & 0 & 1 & y & 0 & -x & 0 & 0 & -\alpha x \\
0 & 0 & 0 & -y & 1 & x & 0 & -\alpha y & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y
\end{bmatrix}
\]

And

\[
[D] = \begin{bmatrix}
D_{11} & D_{12} & 0 \\
D_{12} & D_{22} & 0 \\
0 & 0 & D_{33}
\end{bmatrix}
\]

the usual constitutive matrix

Where:

\[
D_{11} = D_{22} = \frac{E}{(1-\nu^2)}; \quad D_{12} = \frac{\nu E}{(1-\nu^2)}; \quad D_{33} = \frac{E}{2(1+\nu)};
\]

For \([A]\) and \([K_0]\) see the appendix.

3 PATCH - TESTS

3.1 Study of a simple element: dilatation of the element (R4SB2)

This element is subject to an imposed displacement (Fig.2).

<table>
<thead>
<tr>
<th>Node</th>
<th>(U)</th>
<th>(V)</th>
<th>(\sigma_x)</th>
<th>(\sigma_y)</th>
<th>(\tau_{xy})</th>
<th>(\sigma_x)</th>
<th>(\sigma_y)</th>
<th>(\tau_{xy})</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.01</td>
<td>0.12110^7</td>
<td>0.01</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.01</td>
<td>-0.001</td>
<td>0.01</td>
<td>0</td>
<td>-0.001</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

For \(\sigma_x\) and \(\sigma_y\) see the stress calculations.

The results given by the element are perfectly analogous to the exact solution.

3.2 Dilatation of the element

3.2.1 Dilatation of the element in X direction

The objective of this test is to check the rigid body movement and the dilation of the element in X direction (Fig 3).

The displacement of the corresponding force (translation or dilation) is imposed on all nodes except the node n°5. By comparing the displacement provided by the element with that of the theory of plane elasticity, we can check easily if the element passes this patch test.

Two loading cases were considered:

A: 1st loading case: the rigid body movement, the displacement \(U = 10\); for all nodes except the node n°5.

B: 2nd loading case: \(U = 0.01\) displacement of nodes 3, 6 and 9; displacement

\(U = 0.004\) for nodes 2 and 8 \((U = 0\) for nodes 1, 4 and 7)
Table 2: Patch-Test. Loading Case A

<table>
<thead>
<tr>
<th>C.C. node</th>
<th>R4SB2</th>
<th>Theory of Plane elasticity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>-0.4008.10^-2</td>
</tr>
</tbody>
</table>

Table 3: Patch-Test. Loading Case B

<table>
<thead>
<tr>
<th>C.C. node</th>
<th>R4SB2</th>
<th>Theory of Plane elasticity</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.004</td>
<td>-4.0.10^-4</td>
</tr>
</tbody>
</table>

3.2.2 Dilatation of the element Y direction (Fig. 4)

A: 1st loading case: The rigid body movement V=10 displacement for all nodes except the node n°5.

B: 2nd loading case: V=0.01 displacement of nodes 7, 8 and 9; displacement V=0.00333333 for nodes 4 and 6 (V=0 for nodes 1, 2 and 3)

Figure 4: Patch-Test. Dilatation in Y direction

Data : E=1500, ν = 0.2, t =1

Table 4: Patch-Test. Loading Case A

<table>
<thead>
<tr>
<th>C.C. node</th>
<th>R4SB2</th>
<th>Theory of Plane elasticity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>-0.7738.10^-2</td>
<td>10 0 10</td>
</tr>
</tbody>
</table>

Table 5: Patch-Test. Loading Case B

<table>
<thead>
<tr>
<th>C.C. node</th>
<th>R4SB2</th>
<th>Theory of Plane elasticity</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>-0.001333</td>
<td>0.00333333 0.00133333 0.00333333</td>
</tr>
</tbody>
</table>

4 NUMERICAL EXAMPLES

The numerical results of several quadrilateral plane elements is used and compared with those obtained from the present R4SB2 element and they are listed as follows:

- SBRIE: the strain based rectangular in-plane element [4].
- SBRIE2: the strain based rectangular in-plane element with an internal node [4].
- Q4: the standard four-node isoperimetric element.
- Most of the examples dealt with have been proposed at various stages in open literature to validate element performance. It will be seen that the SBRIE and the SBRIE2 versions show the same results for all cases.

4.1 An elongated thin cantilever beam subjected to end shear

An elongated thin cantilever beam subjected to end shear is a standard problem to test finite element accuracy. Young's modulus and Poisson's ratio are denoted by E and ν. These parameters and the mesh division are shown in Fig.5, while the results are presented in Table 6, it should be noted that the R4SB2 element gives the most accurate results.

Figure 5: An elongated thin cantilever beam subjected to end shear

E=1.0x10^7 , ν =0.3 , thickness=0.1

Table 6: Normalized deflection at point A, of a thin cantilever beam under shear

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Normalised tip deflection</th>
</tr>
</thead>
<tbody>
<tr>
<td>1x6</td>
<td>0.903</td>
</tr>
<tr>
<td>SBRIE</td>
<td>0.093</td>
</tr>
<tr>
<td>Q4</td>
<td>0.093</td>
</tr>
<tr>
<td>R4SB2</td>
<td>1.000</td>
</tr>
<tr>
<td>SBRIE2</td>
<td>0.682</td>
</tr>
<tr>
<td>Analyt.</td>
<td>1.000 (0.1081)</td>
</tr>
</tbody>
</table>

4.2 An elongated thin cantilever beam subjected to end pure bending

The tip deflection of an elongated thin cantilever beam under pure bending is compared using the present element R4SB2. The geometry, parameters and mesh discretisation of the beam are shown in Fig.6. Using four different mesh divisions, the normalised tip deflections of the R4SB2 are computed and compared with those obtained by other elements in Table 7. A pertinent point to note is that exact solution can be obtained for the R4SB2 element. The accuracy of the SBRIE and SBRIE2 element is quite high.
produces results which are acceptable within practical engineering accuracy.

**Stresses** $\sigma_{xy}$ at point C:

Table 8: Simply supported beam, Shear stress at point C

<table>
<thead>
<tr>
<th>Mesh</th>
<th>SBRIE 2</th>
<th>Q4</th>
<th>R4SB2</th>
</tr>
</thead>
<tbody>
<tr>
<td>12x4</td>
<td>6.0160</td>
<td>5.2582</td>
<td>6.0344</td>
</tr>
<tr>
<td>12x6</td>
<td>6.0745</td>
<td>5.4544</td>
<td>6.0810</td>
</tr>
<tr>
<td>16x8</td>
<td>6.1606</td>
<td>5.8177</td>
<td>6.1645</td>
</tr>
<tr>
<td>20x10</td>
<td>6.2143</td>
<td>5.9922</td>
<td>6.2172</td>
</tr>
</tbody>
</table>

Exact Solution [2] $\sigma_{xy}(C)=6.3$

**Stresses** $\sigma_{xx}$ at point B:

Table 9: Simply supported beam, Normal Stress at point B

<table>
<thead>
<tr>
<th>Mesh</th>
<th>SBRIE 2</th>
<th>Q4</th>
<th>R4SB2</th>
</tr>
</thead>
<tbody>
<tr>
<td>12x4</td>
<td>28.7085</td>
<td>30.6545</td>
<td>28.7425</td>
</tr>
<tr>
<td>12x6</td>
<td>24.6435</td>
<td>23.8300</td>
<td>24.8365</td>
</tr>
<tr>
<td>16x8</td>
<td>24.2135</td>
<td>24.3630</td>
<td>24.2730</td>
</tr>
<tr>
<td>20x10</td>
<td>24.5055</td>
<td>24.5860</td>
<td>24.5575</td>
</tr>
</tbody>
</table>

Exact Solution [2] $\sigma_{xx}(B)=25.2$

5 CONCLUSION

From the previous examples, the developed element appears to be more accurate and versatile than the standard displacement based element. The robustness of the present element R4SB2 via the patch test has also been shown. The numerical tests demonstrate that satisfactory finite element solutions can be obtained for beam bending without the use of large number of elements.

APPENDIX

Matrices $[A]$ and $[k_0]$ for element R4SB2: with $\alpha=150$

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & a & 0 & -a & 0 & 0 & -a \\
0 & 1 & 0 & a & 0 & 0 & -a & 0 \\
1 & 0 & b & ab & 0 & -b & 0 & -ab \\
0 & 1 & -b & ab & 0 & b & 0 & ab \\
0 & 0 & -\alpha & b & 0 & 0 & -\alpha & 0 \\
0 & 0 & -b & 0 & 0 & -\alpha & 0 & -b \\
\end{pmatrix}
\]

\[
[k_0] = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & a & 0 & -a & 0 & 0 & -a \\
0 & 1 & 0 & a & 0 & 0 & -a & 0 \\
1 & 0 & b & ab & 0 & -b & 0 & -ab \\
0 & 1 & -b & ab & 0 & b & 0 & ab \\
0 & 0 & -\alpha & b & 0 & 0 & -\alpha & 0 \\
0 & 0 & -b & 0 & 0 & -\alpha & 0 & -b \\
\end{pmatrix}
\]
An assumed strain based on triangular element with drilling rotation

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & H_8 & H_9
H_{10} & H_{11} & H_{12} & H_{13} & H_{14} & H_{15} & H_{16} & H_{17} & H_{18}
H_{19} & H_{20} & H_{21} & H_{22} & H_{23} & H_{24}
\end{bmatrix}$$

Where:

$$D_{11} = D_{22} = \frac{E}{(1 - \nu^2)} \quad D_{12} = \frac{\nu E}{(1 - \nu^2)} \quad D_{33} = \frac{E}{2(1 + \nu)}$$

REFERENCES


